PHYSICAL REVIEW E, VOLUME 65, 045604(R)

Amplitude coda of classical waves in disordered media

Mauro S. Ferreira and Gerrit E. W. Bauer
Department of Applied Physics and Delft Institute of Microelectronics and Submicron Technology, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands
(Received 18 January 2002; published 10 April 2002)

The propagation of classical waves in the presence of a disordered medium is studied. We consider wave pulses containing a broad range of frequencies in terms of the configurationally averaged Green function of the system. Damped oscillations in the time-dependent response trailing behind the direct arrival of the pulse (coda) are predicted, the periods of which are governed by the density of scatterers.

DOI: 10.1103/PhysRevE.65.045604 PACS number(s): 46.40.−f

Wave propagation in disordered media is a broad research topic, with many applications. Electron transport in mesoscopic systems, light diffusion in opaque media, or acoustic propagation in the subsurface of the earth are examples of the interdisciplinary character of this general subject. All these examples share a number of common properties, the most important being that they are governed by wave equations. Methods and techniques used in one field of study can, in principle, be applied to others, and such an approach has proved successful in the past. Anderson localization [1], for instance, was originally discovered in the quantum realm of electrons but later understood to be a general wave phenomenon [2–4].

Other contributions from traditionally quantum methods into the study of classical waves are the diagrammatic perturbation technique [5] and the random matrix theory [6]. Whereas the latter explains some universal features in the response of a disordered environment that are independent of the detailed structure of the system, the microscopic nature of the former allows a more specific modeling of the medium. If one wishes to image the medium by observing the signal that is generated by a source and scattered by the system inhomogeneities, the desired information must be sought in the nonuniversal part of the measurements.

In addition to studies of classical wave propagation based on nearly monochromatic sources, the time-dependent response of a short pulse containing a broad range of frequencies is also of considerable interest. This is typically the case in seismic studies of the earth. Excited either by earthquakes or by artificially controlled explosions, seismic waves travel long distances before being registered by an array of detectors. The measurements consist of time-dependent functions describing how the subsurface responds to the excitations. The portion of the wave lagging behind the first arrivals (the so-called coda) is delayed by being repeatedly scattered by interfaces and inhomogeneities and, therefore, carries information about the medium [7]. Only recently it has been realized that the coda of seismic waves can be used for mapping the subsurface of the earth as well as locating earthquake centres [8].

Unlike their quantum counterparts, the amplitude of classical waves can be probed quite trivially. This is the case for seismic and acoustic waves in general. Ultrasonic wave amplitudes are also observable and have been used to investigate multiple scattering effects in disordered media [9,10]. Amplitude and phase of electromagnetic waves have also been measured in the microwave region and used to treat dynamical aspects of the propagation [11,12]. Such a wide applicability motivates us to study the time evolution of the wave amplitude within a disordered medium.

Bearing in mind the microscopic approach mentioned above, we have recently investigated the time response of a disordered medium composed of small spheres (Rayleigh scatterers) [13]. The scatterer is characterized by a set of resonances that appear as poles of the scattering matrix on the lower half of the complex frequency plane. A simple expression for the time-dependent wave amplitude is obtained when all but the lowest order poles are discarded. We subsequently considered an ensemble of randomly placed identical scatterers and related some features of the response to the microscopic details of the structure.

In this paper we extend this idea by accounting for all poles of the scattering matrix. On the one hand, we loose the simplicity of the single pole approximation, but on the other the single scatterer contribution is better described by considering scattering events previously neglected. In spite of the somewhat more complicated formalism, we are still able to relate the delayed-time response to the microscopic parameters of the system in the limit of impenetrable scatterers. Oscillations in the time-dependent amplitude of the wave are identified, the periods of which depend on the concentration of scatterers.

We consider a general scalar wave equation in the frequency domain given by the following eigenvalue equation:

$$\{-\nabla^2+V(\vec{r},\omega)\}\Psi_m(\vec{r},\omega)=E_m(\omega)\Psi(\vec{r},\omega), \tag{1}$$

where $\Psi_m(\vec{r},\omega)$ is the wave field for a frequency ω at position $\vec{r}$. Both the energy E_m and the potential energy V are frequency dependent and given by $V(\vec{r},\omega)=(\omega^2/c_0^2)[1-c_0^2/c^2(\vec{r})]$ and $E_m(\omega)=\omega^2/c_0^2$. The wave velocity $c(\vec{r})$ is position dependent, c_0 being the velocity for the homogeneous background medium. Associated with Eq. (1) there is a Green function G given by

$$G(\vec{r},\vec{r}_0,\omega)=\sum_m \frac{\Psi_m(\vec{r},\omega)\Psi_m^*(\vec{r}_0,\omega)}{(\omega^2/c_0^2)-E_m(\omega)+i\eta\,\mathrm{sgn}(\omega)}, \tag{2}$$

where η is a positive infinitesimal number. $G(\vec{r},\vec{r}_0,\omega)$ represents the response measured at position $\vec{r}$ due to a stationary perturbation of frequency ω produced at $\vec{r}_0$. For the sake of simplicity we assume $\vec{r}_0$ at the origin throughout this paper. The Fourier transform to time domain gives the physically relevant quantity $G(\vec{r},t)$, namely, the response to a pulse containing the full spectrum of frequencies. From this quantity a simple convolution gives the wave amplitude for a pulse of arbitrary shape.

For a disordered system with many scatterers we focus on the configurationally averaged Green function $\langle G(\vec{r},t)\rangle$. In wave vector $\vec{k}$ and frequency space, it is given by

$$\langle G(\vec{k},\omega)\rangle=\frac{1}{(\omega/c_0{}^2)-k^2-\Sigma(\vec{k},\omega)}, \tag{3}$$

where the so-called self-energy $\Sigma(\vec{k},\omega)$ is a complex quantity. The solution to the disordered problem is determined by the behavior of the self-energy. For spherically symmetric scattering cross sections (s-wave) and within the independent scatterer approximation [14], valid for a low concentration of small defects n, the $\vec{k}$ dependence of the self-energy disappears and Σ becomes

$$\Sigma(\omega)=\frac{nc_0(S_0(\omega)-1)}{2i\omega}, \tag{4}$$

where $S_0(\omega)$ is the s-wave scattering matrix for the individual scatterers. Note that Eqs. (3) and (4) bridge the gap between the microscopic scale, in terms of S_0 of a single impurity, and the macroscopic medium, described by the average Green function $\langle G\rangle$.

The scatterers are modeled by small spherical regions within which the waves propagate with velocity c. The corresponding S matrix is given by [15]

$$S_0(\omega)=-e^{-i\omega d/c_0}\left(1+\frac{2i}{(c_0/c)\cot\left(\frac{\omega d}{c}\right)-i}\right), \tag{5}$$

where d is the diameter of the scatterer. S_0 has an infinite number of poles in the frequency domain, each one of them on the lower half of the complex frequency plane. As shown in Ref. [13], a simple expression for the response is obtained when all but the lowest order poles are neglected. Here we want to include all poles of the S matrix and investigate the effect this may bring to the final response. After substituting Eq. (5) into Eq. (4), the Fourier transform of Eq. (3) yields the average Green function $\langle G(\vec{r},t)\rangle$ in position and time domain. Alternatively, we can look at the scattered response $\langle\Delta G(\vec{r},t)\rangle$, defined as the difference in amplitude between the total signal and the impurity-free response, which describes how the scattered portion of the waves evolves in time. Figure 1 shows the calculated $\langle\Delta G(\vec{r},t)\rangle$ for a random medium with identical scatterers at $n=5\times10^{-2}$ and $n=10^{-1}$ volume concentrations and $c=0$ and $c=\frac{1}{5}$. n and c are expressed in units of $(1/d^3)$ and c_0, respectively, whereas the time is in units of d/c_0. The distance r to the source is arbitrarily fixed at $r=10d$. As expected, the direct arrival of the wave is followed by an exponentially decaying response whose features depend on n as well as on the scattering strength. In addition, clear oscillations in the scattered response can be seen in the insets of Fig. 1. Although not shown in the figure, oscillations are also present when $c>c_0$.

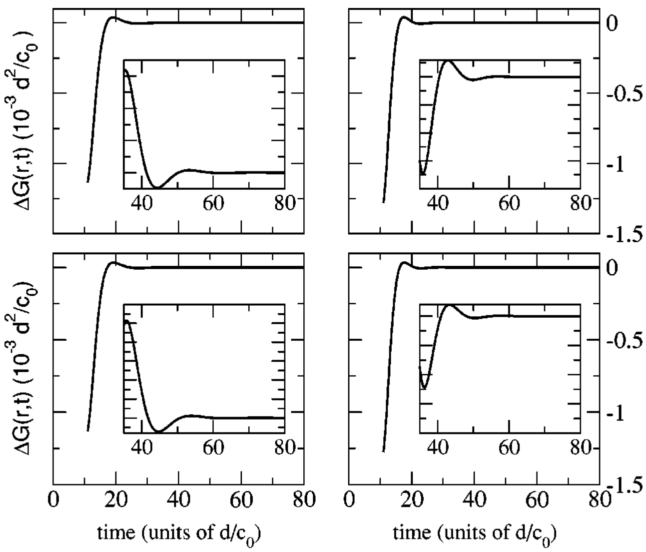

FIG. 1. Scattered response $\langle\Delta G(\vec{r},t)\rangle$ (in units of $10^{-3}\times d^2/c_0$) for $r=10d$. The graphs at the top correspond to the case of impenetrable spheres where $c=0$. The bottom graphs are for $c=c_0/5$. The graphs on the left and right are for $n=5\times10^{-2}$(in units of $1/d^3$) and $n=10^{-1}$ (in units of $1/d^3$), respectively. The insets are amplified by a factor 10.

To understand the origin of those oscillations it is helpful to simplify the problem and assume that the wave velocity vanishes inside the scatterers, i.e., $c=0$. This is equivalent to a set of obstacles that are impenetrable to the waves, modeling, for instance, strong discontinuities in the constitutive parameters. One example is a solid porous medium filled with rarefied air, or any other environment with scatterers of relative low sound velocity. Alternatively, in the case of electromagnetic waves, obstacles with high refractive index are required. With this simplification, the concentration n and the diameter d are the only variable parameters and the S matrix becomes $S_0=-e^{-i\omega d/c_0}$. The response $\langle G(\vec{k},t)\rangle$ in wavenumber and time domain is an intermediate step towards $\langle G(\vec{r},t)\rangle$ and results from the time Fourier transform of Eq. (3), given by

$$\langle G(\vec{k},t)\rangle=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\left(\frac{\omega}{c_0}\right)\times\left\{\frac{e^{-i\omega t}}{(\omega/c_0)^3-(\omega/c_0)k^2+\frac{n}{2i}[e^{-i\omega d/c_0}+1]}\right\}. \tag{6}$$

Although the spatial part of the Fourier transform remains to be done, Eq. (6) is useful for understanding the time dependence of the response. The frequency integral can be evaluated by contour integration. The poles are at k-dependent frequency values that govern the actual dispersion relation of the wave motion. In the absence of scatterers $(n=0)$, for instance, the poles are at $\omega=c_0k$ and a further Fourier transform into space domain gives the usual free-space solution [16] $\mathcal{G}(\vec{r},t)=(c_0/4\pi r)\,\delta(c_0t-r)$. For $n\neq 0$, the poles acquire a concentration dependence $\omega_j(k,n)$, where j is a labeling index. In general, the residue $\mathcal{R}$ associated with a first-order pole $\omega_j(k,n)$ is given by

$$\mathcal{R}[\omega_j(k,n)]=\frac{\omega_j(k,n)e^{-i\omega_j(k,n)t}}{2\pi c_0\phi_j'}, \tag{7}$$

where ϕ_j' stands for the ω derivative of $\phi=(\omega/c_0)^3-(\omega/c_0)k^2+(n/2i)[e^{-i\omega d/c_0}+1]$ evaluated at $\omega=\omega_j(k,n)$. The time dependence of the residue is entirely in the exponential $\exp[-i\omega_j(k,n)t]$. The integral of Eq. (6), written as a sum of residues, is

$$\langle G(\vec{k},t)\rangle=-\frac{i}{c_0}\sum_j\frac{\omega_j(k,n)e^{-i\omega_j(k,n)t}}{\phi_j'}. \tag{8}$$

In this equation each term gives an exponentially damped oscillatory contribution, whose period and rate of decay depend on the real and imaginary parts of $\omega_j(k,n)$, respectively. Whether this time dependence persists in the final response depends on the spatial Fourier transform.

After integrating over the angular variables, the remaining transform becomes

$$\langle G(\vec{r},t)\rangle=\sum_j\int_{-\infty}^{+\infty}dk\mathcal{A}(k,r)e^{-i\omega_j(k,n)t}, \tag{9}$$

where $\mathcal{A}(k,r)=ik^2\sin(kr)\omega_j(k,n)/c_0r\phi_j'$. Except for the damping component induced by the imaginary part of the pole, the integrand in Eq. (9) is a periodic function of t, oscillating with a frequency given by the real part of $\omega_j(k,n)$ and with an amplitude governed by $\mathcal{A}(k,r)$. For large values of t the exponential oscillates rapidly as a function of k and the dominant contribution to the integral comes from the vicinity of k points at which $\omega_j(k,n)$ is stationary. In other words, the values of k that effectively contribute to the integral are those satisfying the condition $d\omega_j/dk=0$. Although each pole in Eq. (6) contributes with a different term in Eq. (9), those meeting the stationary phase condition will dominate the signal.

The existence of such stationary points is implied by the oscillations found in Fig. 1 and we now try to locate them. A closer look at Eq. (6) shows that the poles $\omega_j(k,n)$ are functions of k^2. Therefore, $k=0$ always satisfies the stationary phase condition $d\omega_j/dk=0$. We assume this value of k as a possible candidate for explaining the observed results in Fig. 1. The corresponding poles are the solutions of a transcendental equation, which can be further simplified if we expand the S matrix for small ω. Assuming that n is small, which is the basic condition for the validity of the independent scatterer approximation, the zero order term is sufficient and we find for the real part of the pole that $\mathrm{Re}[\omega_j(k=0,n)]=(c_0\sqrt{3}/2)(n)^{1/3}$. Associated with this pole is a simple expression for the oscillation period T given by

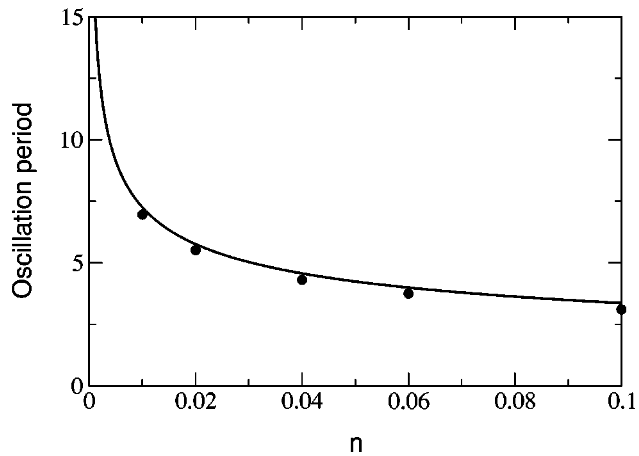

FIG. 2. Oscillation periods (in units of d/c_0) as a function of the concentration n (in units of $1/d^3$). The continuous line results from the stationary phase aproximation whereas the points are the periods obtained after numerical integration.

$$T=2\pi/\mathrm{Re}[\omega_j(k=0,n)]=\left(\frac{4\pi}{c_0\sqrt{3}}\right)\sqrt[3]{1/n}. \tag{10}$$

The predicted oscillation periods based on the stationary phase argument, shown as a continuous line in Fig. 2, agree very well with the observed periods obtained by numerical integration of the Green function, represented by the points. Although the stationary phase approximation provides no information about the respective weight of the oscillations, it is sufficient to explain their existence in the time response $\langle G(\vec{r},t)\rangle$. With such a simple expression for the period, the oscillations are easily related to a microscopic structural parameter of the system, namely the average separation $a=1/\sqrt[3]{n}$ between scatterers.

A simple physical picture for the oscillations in the time-dependent amplitude is as follows. When scattered only once, the S matrix S_0 induces a phase shift in the wave corresponding to a sign change. A second scattering event restores the original sign, which is again changed by a third scatterer, and so on. In this way, positive and negative contributions to the scattered response occur, depending on the number of scattering events. The maximum probability of being scattered a certain number of times depends on the time lapse, and, therefore, dictates the overall sign of the final response. For that reason the sign of the response changes as a function of time and we can also understand why the average distance between scatterers appears as the period of the oscillations. On the opposite side of the velocity spectrum, infinitely large values of c lead to $S_0=e^{-i\omega d/c_0}$, causing no sign change in individual scattering events. In this case, no oscillation should occur and, indeed, are not observed in the numerical solution for this limiting case.

Therefore, according to Eq. (10), it might be possible to obtain the values of a and n, once such oscillations in $\langle G(\vec{r},t)\rangle$ are observed. Whether this structural information can be extracted from an actual measurement depends on the type of disorder in the system. Within the conditions for the validity of the present model, i.e., $\omega d \ll 1$ and $a/d \gg 1$, the oscillations do not depend on the distribution of the size of the scatterers.

Well defined oscillations in the time-delayed response have been recently observed in ultrasonic waves transmitted across a two-dimensional disordered structure [10]. They have a different origin, but we suspect that the superimposed beating pattern reflects the physics discussed here. This hypothesis could be tested by systematic experiments with different concentration of scatterers.

In summary, we have observed oscillations in the time-dependent response of a system composed of randomly placed spherical scatterers. Based on the stationary phase approximation, we have identified the origin of the oscillations and derived a simple expression for the periods in the limit of impenetrable scatterers, whose dependence on the mean separation between obstacles was established. It is suggested that such oscillations should be observable in disordered systems providing an indirect way of measuring the density of scatterers. When the scatterers are penetrable the simple analytical result presented here has to be modified, but can still be useful in the comparison to the numerical results.

We thank Cees Wapenaar for critically reading the manuscript. This work is part of the research program of the "Stichting Technische Wetenschappen" (STW) and the "Stichting Fundamenteel Onderzoek der Materie" (FOM). G.B. acknowledges support by the NEDO program NAME.

[1] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[2] D.S. Wiersma *et al.*, Nature (London) **390**, 671 (1997).
[3] A. Tourin *et al.*, Phys. Rev. Lett. **79**, 3637 (1997).
[4] Ping Sheng, *Introduction to Wave Scattering, Localization, and Mesoscopic Phenomena* (Academic Press, New York, 1995).
[5] M.C.W. van Rossum and Th.M. Nieuwenhuizen, Rev. Mod. Phys. **71**, 313 (1999).
[6] C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
[7] R.K. Snieder, *Scattering and Inverse Scattering in Pure and Applied Science,* edited by R. Pike and P. Sabatier (Academic Press, San Diego, 2002).
[8] H. Sato and M.C. Fehler, *Seismic Wave Propagation and Scattering in the Heterogeneous Earth* (Springer-Verlag, Berlin, 1998).
[9] J.A. Scales and K. van Wijk, Appl. Phys. Lett. **79**, 2294 (2001).
[10] A. Derode, A. Tourin, and M. Fink, Phys. Rev. E **64**, 036605 (2001).
[11] P. Sebbah, O. Legrand, B.A. van Tiggelen, and A.Z. Genack, Phys. Rev. E **56**, 3619 (1997).
[12] P. Sebbah, R. Pnini, and A.Z. Genack, Phys. Rev. E **62**, 7348 (2000).
[13] G.E.W. Bauer, M.S. Ferreira, and C.P.A. Wapenaar, Phys. Rev. Lett. **87**, 113902 (2001).
[14] S. Doniach and E.H. Sondheimer, *Green's Functions for Solid State Physicists* (Imperial College Press, London, 1998).
[15] E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961).
[16] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).